Pascal's Triangle

Pascal's triangle shows the coefficients of each term in the expansion of a binomial of the form $(x+y)^n$.

Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

Each row begins and ends with a 1. The terms in between the 1's are the sum of the two terms directly above their location diagonally to the left and right. The pattern continues on indefinetly although this example only has the first 8 rows. The diagonal lines are not actually part of the triangle they are just put there so it will be easy to see the pattern. They are also helpful when drawing Pascal's triangle.

Look at the triangle. In the 3rd row, 1 2 1, the 2 is the sum of the 1 above it diagonally to its left and to its right. In the 4th row, 1 3 3 1, the 3 on the left is the sum of the 1 and 2 diagonally above it to the left and the right. The same is true for the second 3. In the 5th row, 1 4 6 4 1, the 4 on the left is the sum of the 1 and 3 diagonally above it to its left and right. The 6 is the sum of the 3 and 3 diagonally above it to its left and right. Lastly the 4 is the sum of the 3 and 1 diagonally above it to its left and right. The pattern continues on indefinetly.

An easier way than using Pascal's triangle to find the coefficients of the terms of the expansion is to evaluate the expression $\binom{n}{k-1}$ where n is the exponent on the expansion and k is the number of the term. So if you want the 3rd term in the expansion of $(x+y)^7$ then $n=7$ and $k=3$. You would evaluate $\binom{7}{3-1} = \binom{7}{2} = 21$.

As you can see in the triangle below the 3rd term of the expansion where the top number is 7 is $\binom{7}{2}$

Modified Version of Pascal's Triangle

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$
$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$
$$\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7}$$

Look at the triangle above. Each term is of the form $\binom{n}{k-1}$ where n is the exponent on the expansion and k is the number of the term.

The only problem with this new method is if you don't know how to evaluate an expression like $\binom{n}{k}$. The top and bottom numbers are non-negative integers and the top number is greater than or equal to the bottom number. This is actually the number of combinations of n objects taken k at a time. The expression comes from the study of probability. It represents the expression $\frac{n!}{(n-k)!k!}$ so evaluate this expression to find the value of $\binom{n}{k}$. The other expression you might not be familiar with is $n!$ which is n factorial and means the product of n and all the integers less than n down to zero but not including zero. So $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$. Note that $0! = 1$.

Suppose you are asked to find the coefficient of the 5th term in the expansion of $(x+y)^6$. Since the exponent on the expansion is 6, look in the row that has 6 in the top and find the 5th term in that row which is $\binom{6}{4}$. All you have to do is evaluate $\binom{6}{4}$ and you have the answer. $\binom{6}{4} = 15$. This can be verified by looking in the Pascal triangle on the first page for the 5th term in the 7th row.

Remember this pattern: the top number is the exponent on the expansion, the bottom number is 1 less than the number of the terms position. If you remember this pattern you can just write the term and evaluate it without having to draw Pascal's triangle.

Here are some more examples without the use of Pascal's triangle:

1. What is the 9th term in the expansion of $(x+y)^{12}$? To draw Pascal's triangle to solve this would require that 13 rows are drawn. If this second method is used the only work that needs to be done is to figure out the value of $\binom{12}{8}$. Remember the top number is the exponent on the expansion and the bottom number is one less than the terms position in the expansion. If I need to find the coefficient of the 9th term in an expansion that is raised to the 12th power I need to use 8 on the bottom and 12 on top. $\binom{12}{8}$ evaluates to 495.
2. What is the 10th term in the expansion of $(x+y)^{12}$? The only work that needs to be done is to figure out the value of $\binom{12}{9}$. Remember the top number is the exponent on the expansion and the bottom number is one less than the term number. If I need to find the coefficient of the 10th term in an expansion that is raised to the 12th power I need to use 9 on the bottom and 12 on the top. $\binom{12}{9}$ evaluates to 220.
3. What is the 23th term in the expansion of $(x+y)^{25}$? The only work that needs to be done is to figure out the value of $\binom{25}{22}$. Remember the top number is the exponent on the expansion and the bottom number is one less than the term number. If I need to find the coefficient of the 23rd term in a expansioon that is raised to the 25th power I need to use 22 on the bottom and 25 on the top. $\binom{25}{22}$ evaluates to 300. There is no way I would want to draw Pascal's triangle to figure this one out.

Not only can this method be used to figure out the coefficient of a single term it can be used to figurre out the coefficients of all the terms in a binomial expansion. It is much faster than drawing Pascal's triangle and certainly much faster and less prone to errors than actually expanding the binomial.