Binomial Expansion

Instead of expanding a binomial like $(x+y)^n$ by multiplying $(x+y)$ together n times, the Binomial Expansion method can be used to save steps, time, and reduce the chance of making any errors. Binomial expansion does not get any more complicated as n gets larger but multiplying $(x+y)$ together n times does. As n gets larger and larger, more and more steps are required and the possibility of making an error gets more and more likely.

Here is the formula for Binomial Expansion:

$$(x+y)^n = \sum_{k=0}^{k=n} \binom{n}{k} x^{n-k} y^k$$

You may or may not be familiar with the expression $\binom{n}{k}$ from probability. It means the number of combinations of n objects taken k at a time where k is less than or equal to n. Sometimes $\binom{n}{k}$ is shown as ${}_nC_k$ which means the same thing. The reason $\binom{n}{k}$ is so important here is that it is the value of the coefficient of each term in the expansion when both x and y have coefficients of 1. When x or y or both of them have coefficients other than 1 then $\binom{n}{k}$ is a factor of the coefficient of each term in the expansion.

To evaluate $\binom{n}{k}$ use this formula $\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$. Remember that $n! = 1 \cdot 2 \cdot 3 \cdots \cdot n$ ($n!$ is the product of n and all the integers less than n down to and including 1. So $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$)

Here are some examples to help your understanding.

Example 1

$$(x+y)^1 = \binom{1}{0} x^{1-0} y^0 + \binom{1}{1} x^{1-1} y^1$$

which simplifies to

$$(x+y)^1 = x + y$$

Example 2

$$(x+y)^2 = \binom{2}{0} x^{2-0} y^0 + \binom{2}{1} x^{2-1} y^1 + \binom{2}{2} x^{2-2} y^2$$

which simplifies to

$$(x+y)^2 = x^2 + 2xy + y^2$$

Example 3

$$(x+y)^3 = \binom{3}{0}x^{3-0}y^0 + \binom{3}{1}x^{3-1}y^1 + \binom{3}{2}x^{3-2}y^2 + \binom{3}{3}x^{3-3}y^3$$

which simplifies to

$$(x+y)^3 = x^3 + 3x^2y + 3\mathrm{xy}^2 + y^3$$

Example 4

$$(x+y)^4 = \binom{4}{0}x^{4-0}y^0 + \binom{4}{1}x^{4-1}y^1 + \binom{4}{2}x^{4-2}y^2 + \binom{4}{3}x^{4-3}y^3 + \binom{4}{4}x^{4-4}y^4$$

which simplifies to

$$(x+y)^4 = x^4 + 4x^3y + 6\mathrm{x}^2y^2 + 4\mathrm{xy}^3 + y^4$$

Example 5

$$(x-y)^3 = \binom{3}{0}x^{3-0}(-y)^0 + \binom{3}{1}x^{3-1}(-y)^1 + \binom{3}{2}x^{3-2}(-y)^2 + \binom{3}{3}x^{3-3}(-y)^3$$

which simplifies to

$$(x-y)^3 = x^3 - 3x^2y + 3\mathrm{xy}^2 - y^3$$

Example 6

$$(2\mathrm{x}+y)^4 = \binom{4}{0}(2\mathrm{x})^{4-0}y^0 + \binom{4}{1}(2\mathrm{x})^{4-1}y^1 + \binom{4}{2}(2\mathrm{x})^{4-2}y^2 + \binom{4}{3}(2\mathrm{x})^{4-3}y^3 + \binom{4}{4}(2\mathrm{x})^{4-4}y^4$$

which simplifies to

$$(2\mathrm{x}+y)^4 = (2\mathrm{x})^4 + 4(2\mathrm{x})^3y + 6(2\mathrm{x})^2y^2 + 4(2\mathrm{x})y^3 + y^4$$

and then to

$$(2\mathrm{x}+y)^4 = 16\mathrm{x}^4 + 32\mathrm{x}^3y + 24\mathrm{x}^2y^2 + 8\mathrm{xy}^3 + y^4$$

Example 7

$$(2\mathrm{x}-y)^4 = \binom{4}{0}(2\mathrm{x})^{4-0}(-y)^0 + \binom{4}{1}(2\mathrm{x})^{4-1}(-y)^1 + \binom{4}{2}(2\mathrm{x})^{4-2}(-y)^2 + \binom{4}{3}(2\mathrm{x})^{4-3}(-y)^3 + \binom{4}{4}(2\mathrm{x})^{4-4}(-y)^4$$

which simplifies to

$$(2\mathrm{x}+y)^4 = (2\mathrm{x})^4 - 4(2\mathrm{x})^3y + 6(2\mathrm{x})^2y^2 - 4(2\mathrm{x})y^3 + y^4$$

and then to

$$(2\mathrm{x}-y)^4 = 16\mathrm{x}^4 - 32\mathrm{x}^3y + 24\mathrm{x}^2y^2 - 8\mathrm{xy}^3 + y^4$$

Example 8

$$(2\mathrm{x}-3\mathrm{y})^4 = \binom{4}{0}(2\mathrm{x})^{4-0}(-3\mathrm{y})^0 + \binom{4}{1}(2\mathrm{x})^{4-1}(-3\mathrm{y})^1 + \binom{4}{2}(2\mathrm{x})^{4-2}(-3\mathrm{y})^2 + \binom{4}{3}(2\mathrm{x})^{4-3}(-3\mathrm{y})^3 + \binom{4}{4}(2\mathrm{x})^{4-4}(-3\mathrm{y})^4$$

which simplifies to

$$(2\mathrm{x}-3\mathrm{y})^4 = (2\mathrm{x})^4 + 4(2\mathrm{x})^3(-3\mathrm{y}) + 6(2\mathrm{x})^2(-3\mathrm{y})^2 + 4(2\mathrm{x})(-3\mathrm{y})^3 + (-3\mathrm{y})^4$$

and then to

$$(2\mathrm{x}-3\mathrm{y})^4 = 16\mathrm{x}^4 - 96\mathrm{x}^3y + 216\mathrm{x}^2y^2 - 216\mathrm{xy}^3 + 81\mathrm{y}^4$$

Example 9

$$(-2x-3y)^4 = \binom{4}{0}(-2x)^{4-0}(-3y)^0 + \binom{4}{1}(-2x)^{4-1}(-3y)^1 + \binom{4}{2}(-2x)^{4-2}(-3y)^2 + \binom{4}{3}(-2x)^{4-3}(-3y)^3 + \binom{4}{4}(-2x)^{4-4}(-3y)^4$$

which simplifies to

$$(-2x-3y)^4 = (-2x)^4 + 4(-2x)^3(-3y) + 6(-2x)^2(-3y)^2 + 4(-2x)(-3y)^3 + (-3y)^4$$

and then to

$$(2x-3y)^4 = 16x^4 + 96x^3 y + 216x^2 y^2 + 216xy^3 + 81y^4$$

Example 10

$$(-2x+3y)^4 = \binom{4}{0}(-2x)^{4-0}(3y)^0 + \binom{4}{1}(-2x)^{4-1}(3y)^1 + \binom{4}{2}(-2x)^{4-2}(3y)^2 + \binom{4}{3}(-2x)^{4-3}(3y)^3 + \binom{4}{4}(-2x)^{4-4}(3y)^4$$

which simplifies to

$$(-2x+3y)^4 = (-2x)^4 + 4(-2x)^3(3y) + 6(-2x)^2(3y)^2 + 4(-2x)(3y)^3 + (3y)^4$$

and then to

$$(2x-3y)^4 = 16x^4 - 96x^3 y + 216x^2 y^2 - 216xy^3 + 81y^4$$