How to find the inverse of a matrix

The inverse of a matrix is found using this formula where A^{-1} is sthe inverse of matrix A :

$$A^{-1}=\frac{1}{det\ A}adj(A)$$

The first step is to calculate $det\ A$ (which is the determinant of A). Be sure that it is not zero because if it is zero than A is a singular matrix and does not have an inverse (A^{-1} does not exist).

The next step is to calculate $adj(A)$ which is a new matrix called the adjoint of A but to do so you need to calculate C the cofactor matrix of A first. Once you have C you transpose the rows and columns of C to create $adj(A)$ and you are ready to find A^{-1}.

Here is an example to help with your understanding of finding the inverse of a matrix.

$A=\begin{bmatrix} 4 & 0 & 5 \\ -3 & -2 & -1 \\ 1 & 2 & 3 \end{bmatrix}$ and $det\ A=\begin{vmatrix} 4 & 0 & 5 \\ -3 & -2 & -1 \\ 1 & 2 & 3 \end{vmatrix}$ which evaluates to -36 so the matrix has an inverse and is not singular.

How to calculate the determinant of a matrix
In order to understand how to create C I need to explain how to calculate $det\ A.$ I used expansion by minors along the top row although any row or column will do. In that process I had to multiply three things together for each element in the top row of $det\ A.$ I had to multiply the element $det\ A(i,j)$ together with its sign element $(-1)^{i+j}$ which is simply (-1) raised to the sum of the row and column that the element $det\ A(i,j)$ is in. Next I had to multiply that product by the determinant of the matrix that resulted when the row and column that contain $det\ A(i,j)$ are removed from $det\ A.$ The resulting matrix is the minor of the element $det\ A(i,j).$ I had to do this for each element in the top row and then add those together to get the value of the determinant. Look at the calculations below for a better understanding.

The calculation of $det\ A$ follows:

$$det\ A(1,1)(-1)^2\begin{vmatrix} -2 & -1 \\ 2 & 3 \end{vmatrix} = 4\cdot 1\cdot(-4)=-16 \ , \quad det\ A(1,2)(-1)^3\begin{vmatrix} -3 & -1 \\ 1 & 3 \end{vmatrix} = 0\cdot(-1)\cdot(-8)=0$$

$$det\ A(1,3)(-1)^4\begin{vmatrix} -3 & -2 \\ 1 & 2 \end{vmatrix} = 5\cdot 1\cdot(-4)=-20$$

Adding the values together gives -36 which is the determinant.

How to get the cofactor matrix $C.$

Each element $det\, A(i,j)$ in $det\, A = \begin{vmatrix} 4 & 0 & 5 \\ -3 & -2 & -1 \\ 1 & 2 & 3 \end{vmatrix}$ has a corresponding element $C(i,j)$ that is the product of the sign factor of the element and the minor of the element $det\, A(i,j).$ The sign factor is given by $(-1)^{i+j}$ where i is the row number of the element and j is the column number of the element. Each element of $C.$ needs to be calculated.

Here are the calculations for all of the elements of $C.$

$$C(1,1) = (-1)^2 \cdot minor\ det\, A(1,1) = (-1)^2 \cdot \begin{vmatrix} -2 & -1 \\ 2 & 3 \end{vmatrix} = 1 \cdot (-4) = -4$$

$$C(1,2) = (-1)^3 \cdot minor\ det\, A(1,2) = (-1)^3 \cdot \begin{vmatrix} -3 & -1 \\ 1 & 3 \end{vmatrix} = -1 \cdot (-8) = 8$$

$$C(1,3) = (-1)^4 \cdot minor\ det\, A(1,3) = (-1)^4 \cdot \begin{vmatrix} -3 & -2 \\ 1 & 2 \end{vmatrix} = 1 \cdot (-4) = -4$$

$$C(2,1) = (-1)^3 \cdot minor\ det\, A(2,1) = (-1)^3 \cdot \begin{vmatrix} 0 & 5 \\ 2 & 3 \end{vmatrix} = -1 \cdot (-10) = 10$$

$$C(2,2) = (-1)^4 \cdot minor\ det\, A(2,2) = (-1)^4 \cdot \begin{vmatrix} 4 & 5 \\ 1 & 3 \end{vmatrix} = 1 \cdot 7 = 7$$

$$C(2,3) = (-1)^5 \cdot minor\ det\, A(2,3) = (-1)^5 \cdot \begin{vmatrix} 4 & 0 \\ 1 & 2 \end{vmatrix} = -1 \cdot 8 = -8$$

$$C(3,1) = (-1)^4 \cdot minor\ det\, A(3,1) = (-1)^4 \cdot \begin{vmatrix} 0 & 5 \\ 2 & -1 \end{vmatrix} = 1 \cdot 10 = 10$$

$$C(3,2) = (-1)^5 \cdot minor\ det\, A(3,2) = (-1)^5 \cdot \begin{vmatrix} 4 & 5 \\ -3 & -1 \end{vmatrix} = -1 \cdot 11 = -11$$

$$C(3,3) = (-1)^6 \cdot minor\ det\, A(3,3) = (-1)^6 \cdot \begin{vmatrix} 4 & 0 \\ -3 & -2 \end{vmatrix} = 1 \cdot (-8) = -8$$

Therefore $C = \begin{bmatrix} -4 & 8 & -4 \\ 10 & 7 & -8 \\ 10 & -11 & -8 \end{bmatrix}$

How to create $adj(A)$ (transpose C)
To create $adj(A)$ make a new matrix where the rows of C are used as the columns of $adj(A)$. That means that row 1 of C will be column 1 of $adj(A)$, row 2 of C will be column 2 of $adj(A)$. , and so on.

$$adj(A) = \begin{bmatrix} -4 & 10 & 10 \\ 8 & 7 & -11 \\ -4 & -8 & -8 \end{bmatrix}$$